Toulouse Complex Network
4-day workshop
May 1-4, 2016

Institut des Systèmes Complexes de Toulouse

Hereditary properties

k-dismantlability

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INSTITUT de MATHEMATIQUES de TOULOUSE

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FRAMESPA

Boolean functions and query complexity

Given a boolean function

$$F : \{0,1\}^n \to \{0,1\}$$

and given a $\sigma = (x_1, x_2, \ldots, x_i) \in \{0,1\}^n$, we want to know the value of $F(\sigma)$, the only questions possible being《what is the value of x_i ?》?

$D(F)$ is the minimum number of questions which permits to know the value of $F(\sigma)$ for every σ.

We have $D(F) = 0$ (if F is a trivial function) or

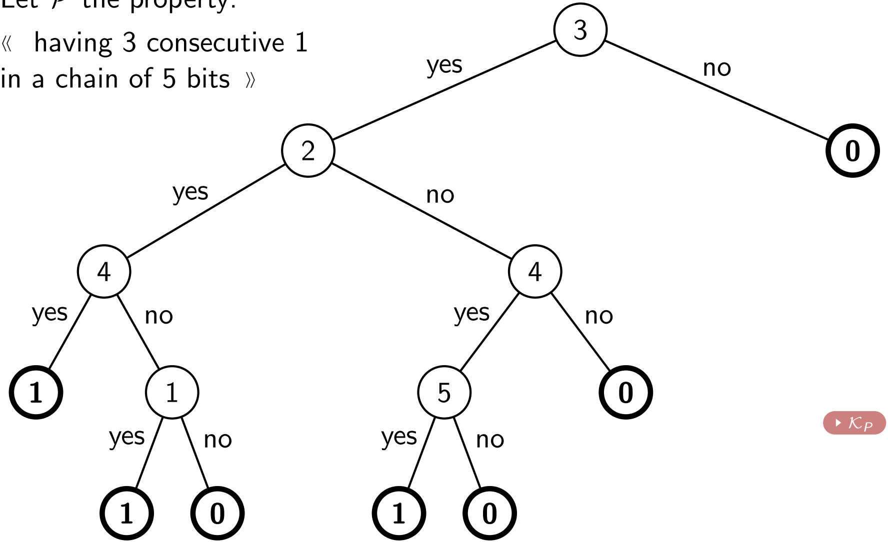
$$1 \leq D(F) \leq n$$

The function F is said evasive if $D(F) = n$ (maximal《complexity》)

Decision tree (example)

Let $\mathcal{P}$ the property:

« having 3 consecutive 1 in a chain of 5 bits »



$\mathcal{K}_P$

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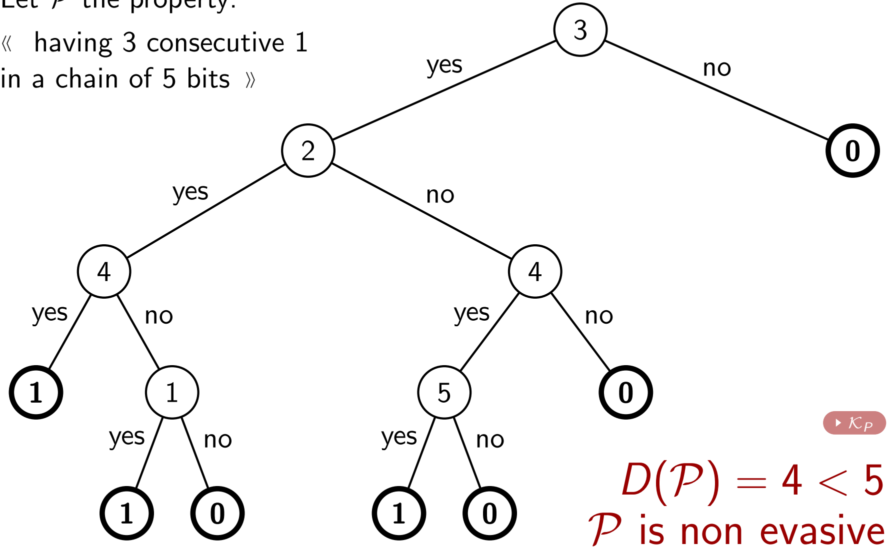


▸ $\mathcal{K}_P$

$D(\mathcal{P}) = 4 < 5$

$\mathcal{P}$ is non evasive

Boolean functions and evasiveness, examples

- « $F(x) = x_1 + x_2 + \ldots + x_n \mod 2$» is evasive.
- $(n = N^2)$: « $f\left((x_{ij})_{1 \leq i,j \leq N}\right) = \bigwedge_i \bigvee_j x_{ij}$» is evasive
- « avoir au moins k 1 » is evasive if, and only if, $1 \leq k \leq n$
- For $n \geq 3$, the property « to have three consecutive 1 » is evasive if, and only if, $n \equiv 0$ or $n \equiv 3$ modulo 4

Graph properties

Let $\mathcal{V}$ a set of k elements. If $n = \binom{k}{2} = \frac{k(k-1)}{2}$, every $\mathcal{E} \subset 2^{[n]}$ represents a graph with vertex set $\mathcal{V}$ and edge set $\mathcal{E}$ ($x_i = 1$ denotes the presence of the edge x_i).

A *graph property* is a set of graphs $\mathcal{P}$ such that

$$(S, A) \simeq (S, A') \Longrightarrow A, A' \in \mathcal{P} \ \text{ ou } \ A, A' \notin \mathcal{P}$$

or may be seen as a boolean function

$$f : \{x_{ij},\ 1 \leq i < j \leq n\} \longrightarrow \{0, 1\}$$

such that, for all permutation $\sigma \in \mathcal{S}_n$:

$$f\left((x_{ij})_{1 \leq i < j \leq n}\right) = f\left((x_{\sigma(i)\sigma(j)})_{1 \leq i < j \leq n}\right)$$

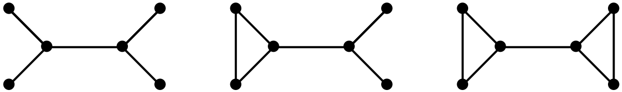
Graph properties and evasiveness

Evasive Proprerties

- being planar (with $n \geq 5$ vertices...)
- having at most j edges with $j < \binom{k}{2}$
- being acyclic
- being connected

Non evasive properties

- ($k = 6$) to be one of the three following graphs :



- to be a tournament with k vertices and a source ($c(P) \leq 3k - 4$)
- to be a *scorpion-graph* with $k \geq 5$ vertices ($c(P) \leq 6k - 13$)

Conjectures about hereditary properties

A graph property $\mathcal{P}$ is *monotone increasing* if it is preserved by addition of edges, i.e. $G = (S, A)$ verifies $\mathcal{P} \Longrightarrow G - e := (S, A - \{e\})$ verifies $\mathcal{P}$ (and *monotone decreasing* if it is preserved by deletion of edges).

Aanderaa-Karp-Rosenberg conjecture

Any *monotone* and non trivial graph property is evasive.

$F : \{0,1\}^n \to \{0,1\}$ is *weakly symmetric* if there is a subgroup Γ of S_n transitive on $\{1, 2, \dots, n\}$ such that F is Γ-invariant, i.e. for all $g \in \Gamma$ and all $(x_i)_{1 \le i \le n} \in \{0,1\}^n$, $F\big((x_i)_{1\le i\le n}\big) = F\big((x_{g(i)})_{1\le i\le n}\big)$.

Generalized AKR conjecture

others versions

Any *monotone*, non trivial and weakly symmetric boolean function is evasive.

Simplicial complexes

An abstract simplicial complex $K = (V(K), \Sigma(K))$ is given by :

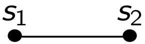
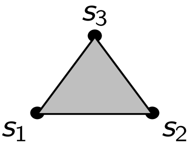
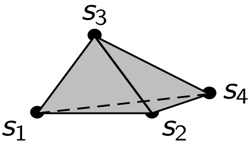
- $V(K)$, a set of vertices
- $\Sigma(K) \subset 2^{V(K)}$ a set of simplices such that

$$\tau \subset \sigma \text{ and } \sigma \in \Sigma(K) \Rightarrow \tau \in \Sigma(K)$$

DÉFINITIONS/NOTATIONS :

- If $\tau \subset \sigma \in \Sigma(K)$ and $\tau \neq \sigma$, one says that τ is a *face* of σ.

- $|K|$: *geometric realization* of K (if $\#V(K) = n$, $|K| \subset \mathbf{R}^n$).

	abstract simplices	geometrical simplices
0-simplex	$\{s_1\}$	
1-simplex	$\{s_1, s_2\}$	
2-simplex	$\{s_1, s_2, s_3\}$	
3-simplex	$\{s_1, s_2, s_3, s_4\}$	

From monotone boolean functions to simplicial complexes

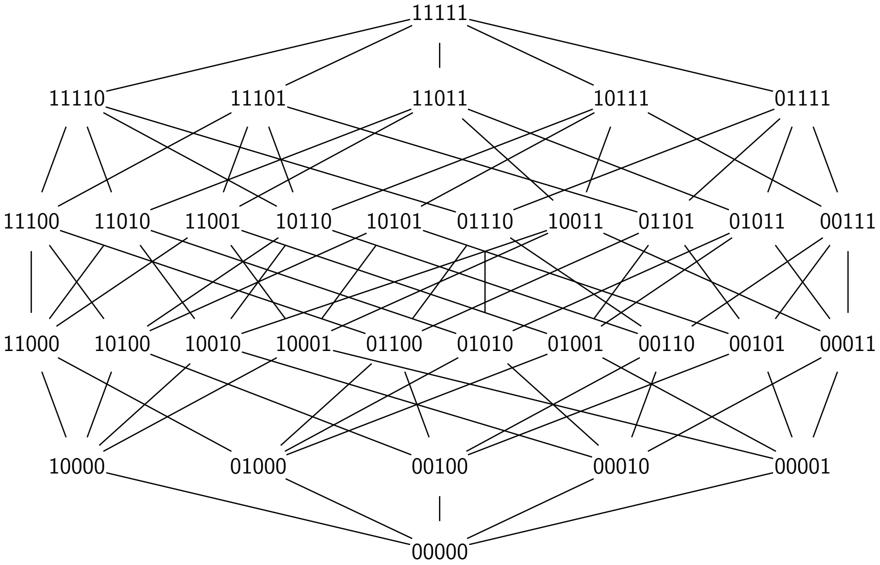
A *monotone increasing* boolean function F defines a simplicial complex

$$\mathcal{K}_F := \{S \subset \{1, \ldots, n\},\ F(x^S) = 0\}$$

where $x_i^S = 1$ if and only if $i \in S$.

Reciprocally, a simplicial complex K defines a monotone boolean function F_K such that $K = \mathcal{K}_{F_k}$ and $K = \mathcal{K}_F$ is said evasive if, and only if, F is evasive

Example with F associated to « (at least) 3 consecutive 1 »



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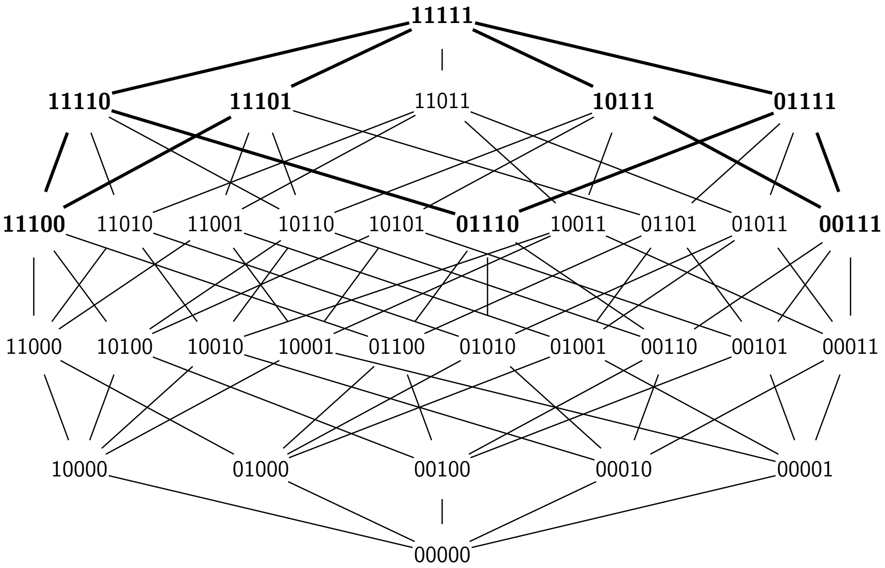

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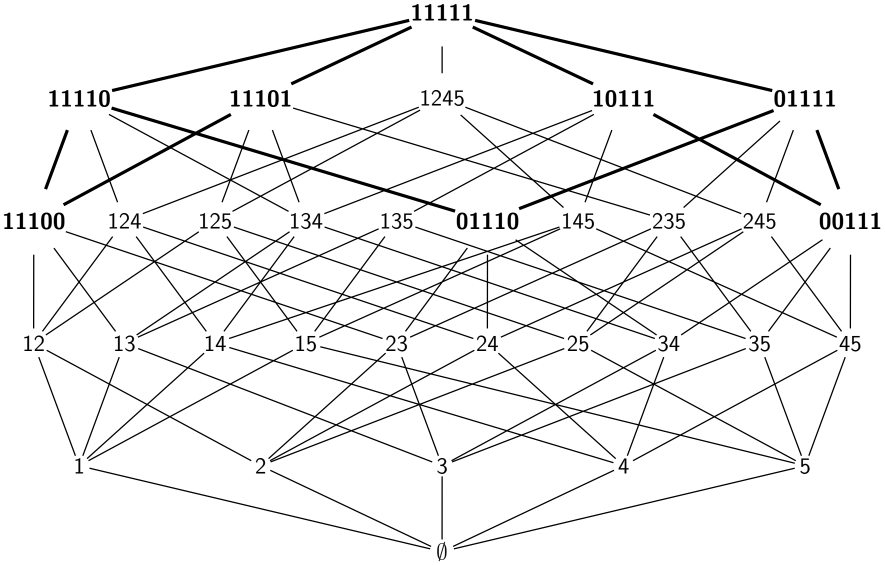
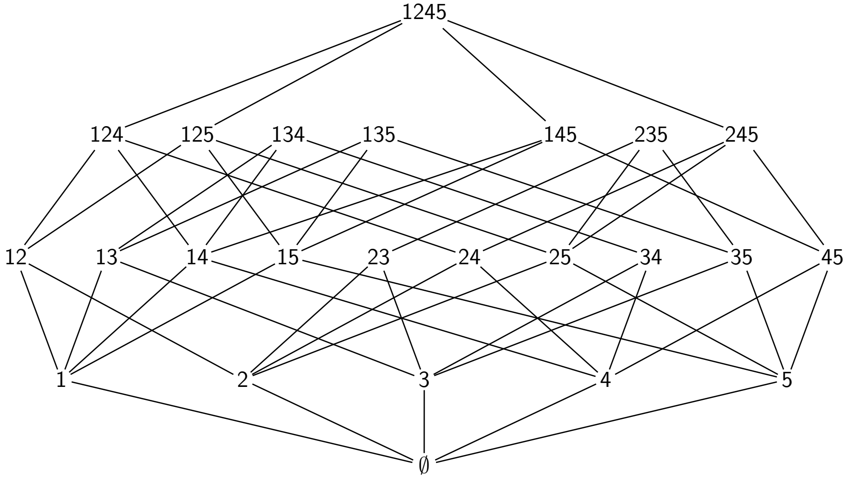


(face poset of the) simplicial complex $\mathcal{K}_F$ obtained for « F : 3 consecutive 1 » :



Why simplicial complexes ?

So, for the evasiveness question, one can replace the monotone boolean function F by the simplicial complex $\mathcal{K}_F$.

And... ?

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And... ?

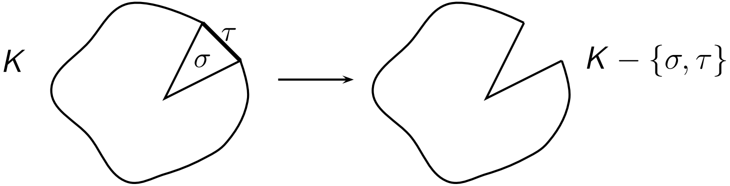
Theorem (Kahn, Saks, Sturtevant, 1984)

A non evasive complex is collapsible.

(recall that : collapsible $\Longrightarrow$ contractible)

Collapsibility

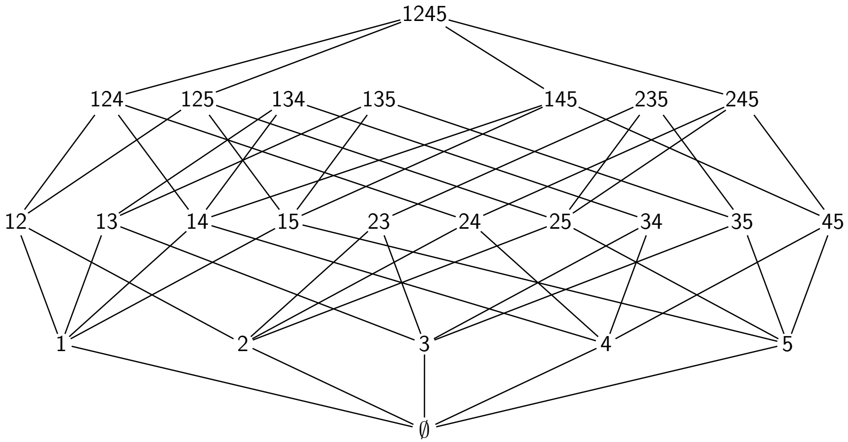
If τ is a maximal face of σ and is not a strict face of another simplex, one says that τ is free face and that $\{\sigma, \tau\}$, is a collapsible pair.



- The deletion of a collapsible pair is an elementary simplicial collapse
- Notation : $K \searrow^{sc} K - \{\sigma, \tau\}$
- A collapse $K \searrow^{sc} L$ is a succession of elementary collapses transforming K in L
- K is said collapsible if $K \searrow^{sc} pt$ where pt is a simplicial complex reduced to a point

Simplicial collapsing of $\mathcal{K}_F$:

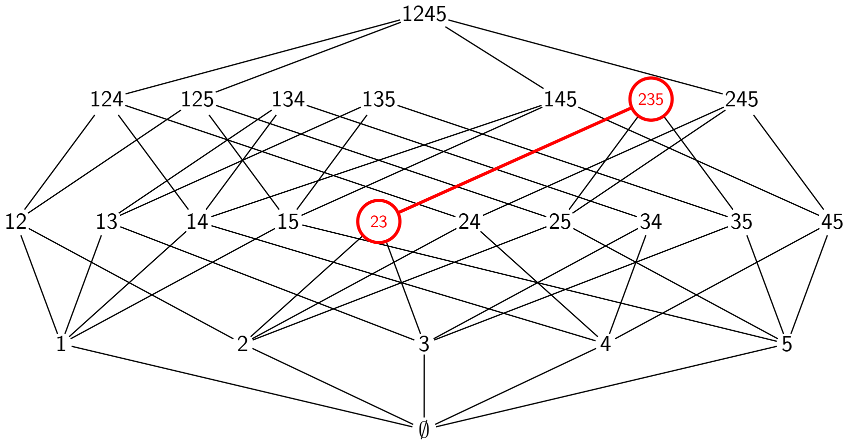
decision tree



Simplicial collapsing of $\mathcal{K}_F$:

decision tree

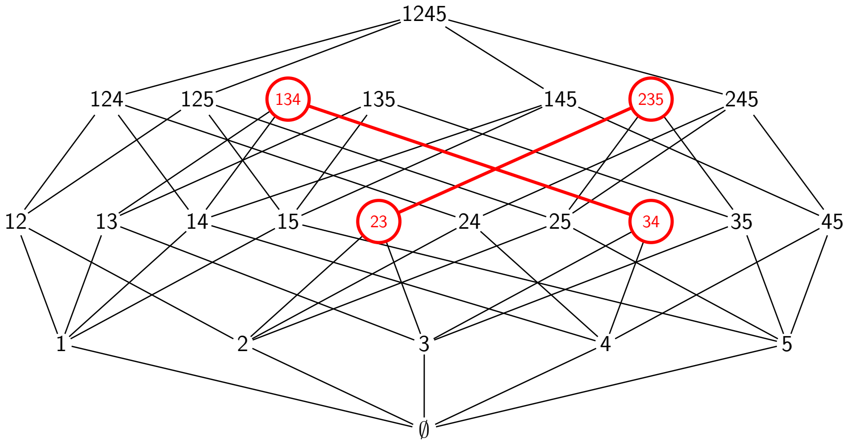
$$\mathcal{K}_F \quad \searrow^{sc} \quad \mathcal{K}_1 := \mathcal{K}_F - \{23, 235\}$$



Simplicial collapsing of $\mathcal{K}_F$:

decision tree

$$\mathcal{K}_F \searrow^{sc} \mathcal{K}_1 := \mathcal{K}_F - \{23, 235\}$$
$$\searrow^{sc} \mathcal{K}_2 := \mathcal{K}_1 - \{34, 134\}$$



Simplicial collapsing of $\mathcal{K}_F$:

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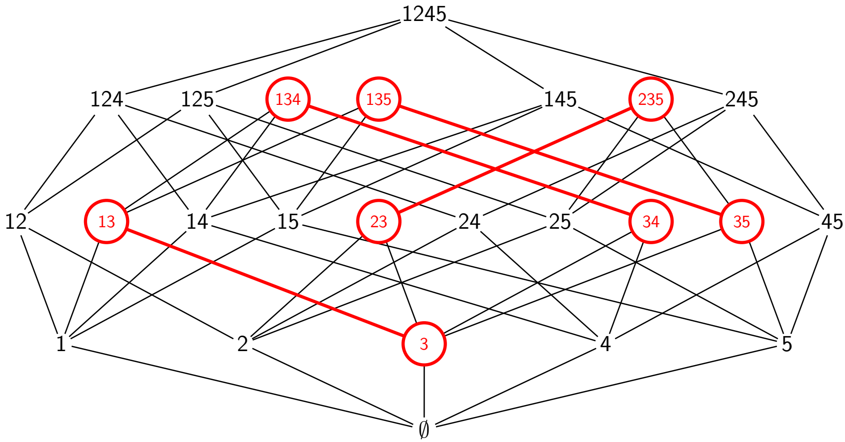
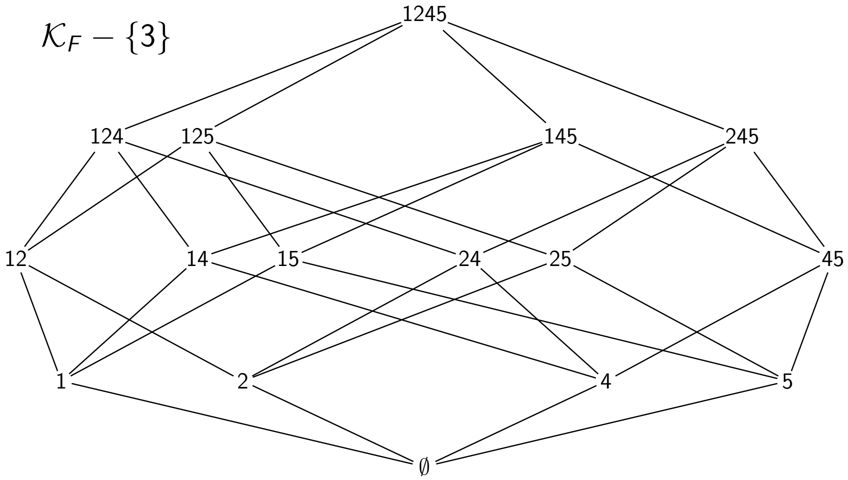
$$\mathcal{K}_F \searrow^{sc} \mathcal{K}_1 := \mathcal{K}_F - \{23, 235\}$$
$$\searrow^{sc} \mathcal{K}_2 := \mathcal{K}_1 - \{34, 134\}$$
$$\searrow^{sc} \mathcal{K}_2 - \{34, 134\} - \{3, 13\} = K - \{3\}$$



Simplicial collapsing of $\mathcal{K}_F$: $\mathcal{K}_F \searrow^{sc} \mathcal{K}_F - \{3\} \searrow^{sc} pt$

$\mathcal{K}_F - \{3\}$



important : collapsible $\Longrightarrow$ contractible

The topological space X is said *contractible* il there is a contiunous map $H : X \times [0,1] \to X$ such that, for all x in X, $H(x,0) = x$ and $H(x,1) = x_0$ for some point x_0 of X.

Brouwer theorem
Let K a simplicial complex and $\varphi : |K| \to |K|$ continue.

$$|K| \ \text{contractile} \quad \Longrightarrow \quad \text{fix}(\varphi) \neq \emptyset$$

where $\text{Fix}(\varphi) := \{x \in |K|, \ \varphi(x) = x\}$ is the set of fixed points of φ.

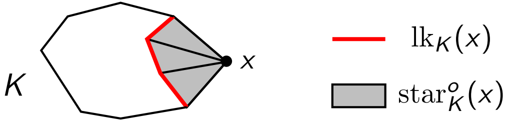
NOTE : Every simplicial $f : K \to K$ (i.e. $f(\sigma)$ is a simplex for any simplex σ), induces $\varphi := |f| : |K| \to |K|$. Nevertheless

$$|K| \ \text{contractible} \ \not\Longrightarrow \ \ \text{Fix}(f) \neq \emptyset$$

non evasiveness of $F \Longrightarrow$ collapsibility of $\mathcal{K}_F$

Let K a simplicial complex and x a vertex of K

- $\mathrm{lk}_K(x) := \{\sigma \in K,\ \{x\} \cup \sigma \in K \text{ et } x \notin \sigma\}$
- $K - x := \{\sigma \in K,\ x \notin \sigma\}$



$$\mathcal{K}_{F|_{x_i=0}} := \{S \subset \{1,\dots,i-1,i+1,\dots,n\},\ F(x^S) = 0\} = \mathcal{K}_f - i$$

$$\mathcal{K}_{F|_{x_i=1}} := \{S \subset \{1,\dots,i-1,i+1,\dots,n\},\ S \cup \{i\} \in \mathcal{K}_F\} = \mathrm{lk}_{\mathcal{K}_F}(i)$$

Proposition The complex K is non evasive if, and only if, it admits a vertex x such that $K - x$ and $\mathrm{lk}_{\mathcal{K}_F}(x)$ are non evasive.

vertex-collapsibilities (J. Barmak, G. Minian 2009)

Let K a simplicial complex and x a vertex (i.e. a 0-simplex) of K

- An elementary strong collapse, denoted $K \searrow\searrow K - x$, is the deletion of a vertex x s.t. $\mathrm{lk}_K(x)$ is a cone. Such a vertex is called 0-collapsible and $Coll_0(K)$ is the set of 0-collapsible vertices of K.
- A strong collapse, denoted $K \searrow\searrow L$, is the succession of elementary strong collapses.
- K is strong collapsible if $K \searrow\searrow pt$ and $Coll_0$ is the set of strong collapsible finite complexes
- A vertex is k-collapsible if $\mathrm{lk}_K(x) \in Coll_{k-1}$
- A complex K is k-collapsible if one can reduce it to a vertex by deleting k-collapsible vertices and $Coll_k$ is the set of strong collapsible finite complexes.

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Proposition : $NE = \bigcup_k Coll_k$ (where NE denotes the set of non evasive simplicial complexes)

Evasiveness conjecture

A simplicial complex K is said vertex-homogeneous if $\mathrm{Aut}(K)$, the group of simplicial automorphisms of K, acts transitively on the vertices of K.

In the framework of simplicial complexes, we get the following reformulations of the generalized AKR conjecture:

Generalized AKR conjecture, version 2

If K is a non evasive and vertex homogeneous simplicial complex, then K is a simplex.

Generalized AKR conjecture, version 3

If $K \in \mathit{Coll}_k$ for some integer $k \geq 0$ and if K is vertex homogeneous, then K is a simplex.

0-dismantlability (« classical » dismantlability)

Graphs $G = (V(G), E(G))$ are finite and reflexive.

A vertex $a \in V(G)$ is called 0-dismantlable if there is another vertex $b \in V(G)$ such that every neighbour of x is also a neighbour of x :

$$N_G[a] \subset N_G[b] \qquad (notation : a \vdash b\,)$$

Then, we say that G is 0-dismantlable on $G - a$; notation : $G \searrow^0 G - a$.

A graph G is called 0-dismantlable if $V(G) = \{x_1, x_2, \ldots, x_n\}$ with

$$G = G_1 \searrow^0 G_2 \searrow^0 G_3 \ldots \searrow^0 G_i \searrow^0 G_{i-1} \searrow^0 \ldots \searrow^0 G_n = \{x_n\}$$

where G_i is the subgraph of G induced by $\{x_i, x_{i+1}, \ldots, x_n\}$.

« cop and rob » game

- Player 1 (the cop) chooses a vertex
- Then, player 2 (the robber) chooses a vertex
- Then, cop and rob move to an adjacent vertex alternatively (first cop, next rob) ; and so on...
- The cop wins if he « catches» the robber (they are on the same vertex)

Theorem (Quilliot 1978, Nowakowski, Winkler 1983, ...)

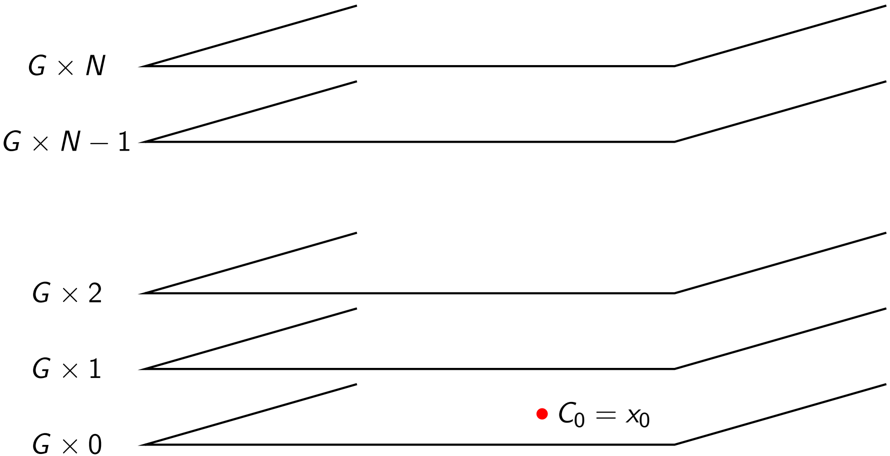
Let G be a reflexive finite graph

G is cop-win $\iff$ G is dismantlable $\iff$ G is contractible

$\mathcal{H} : G \times I_N \to G$ from $\mathcal{H}_0 = C_{x_0}$ to $\mathcal{H}_N = 1_G$ gives a **winning strategy** :

	R_1	R_2	$\dots$	R_{N-1}	R_N
C_0	C_1	C_2	$\dots$	C_{N-1}	C_N

with $C_i = \mathcal{H}_i(R_i)$, for $i = 1, \dots, N$.



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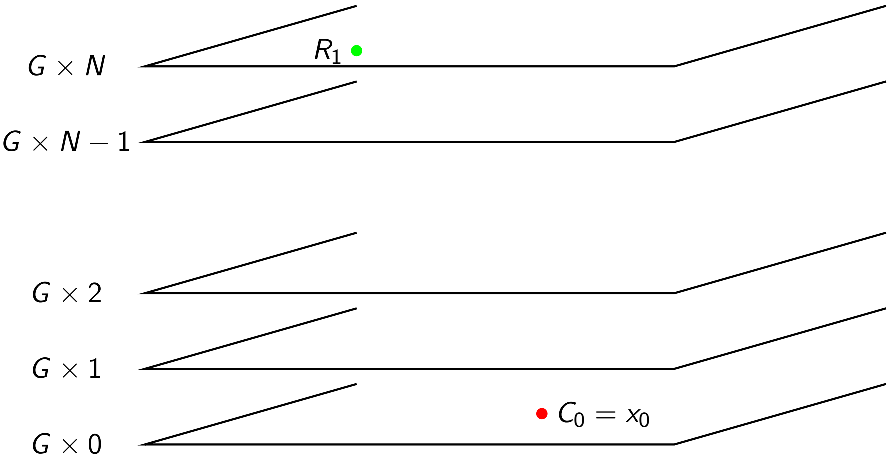

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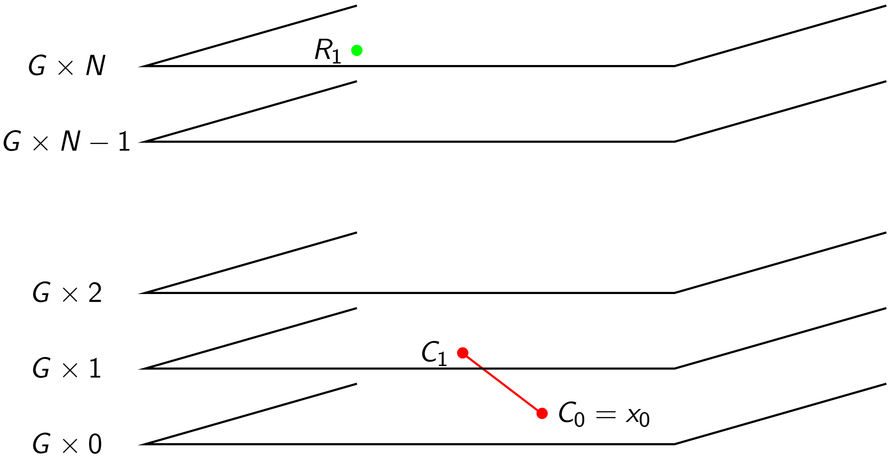

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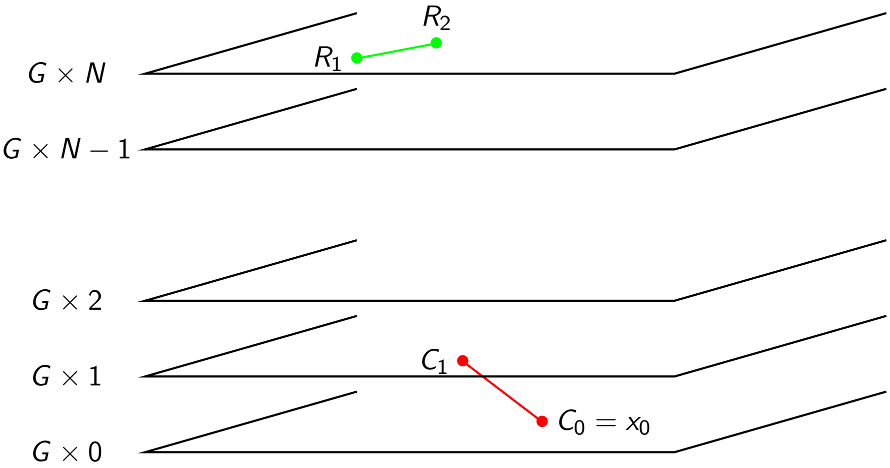

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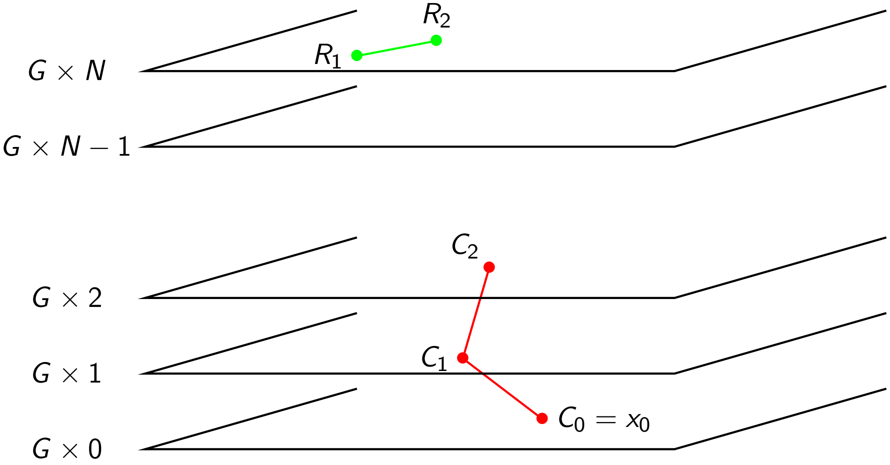

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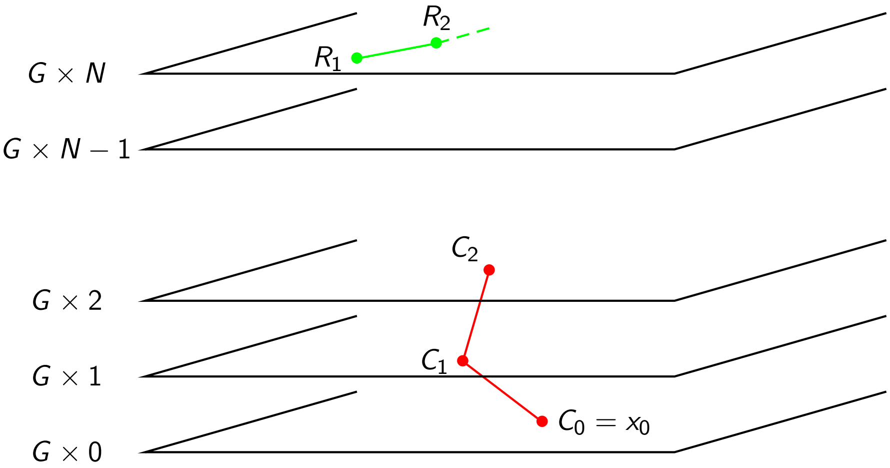

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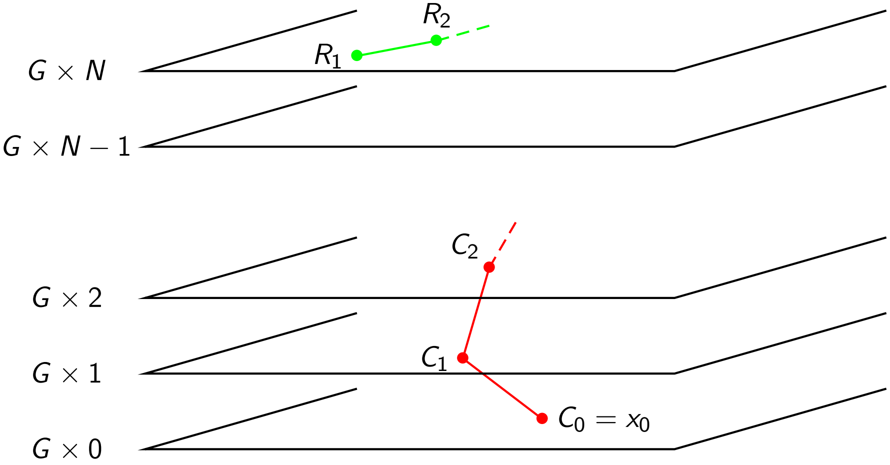

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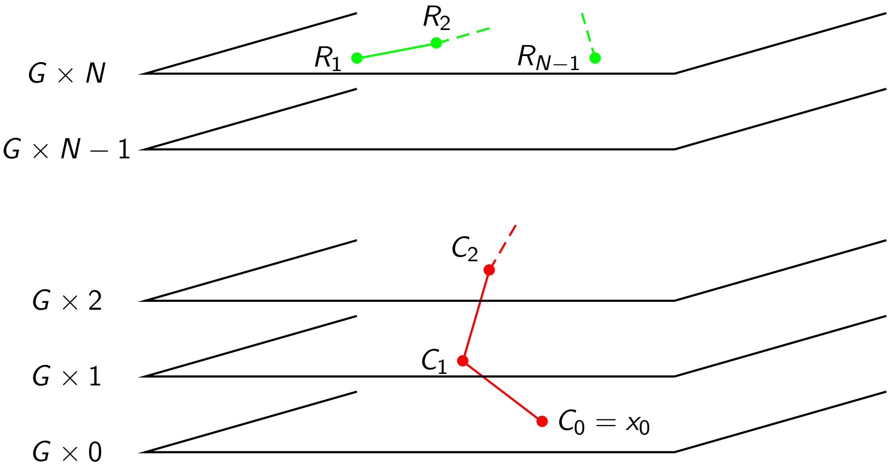

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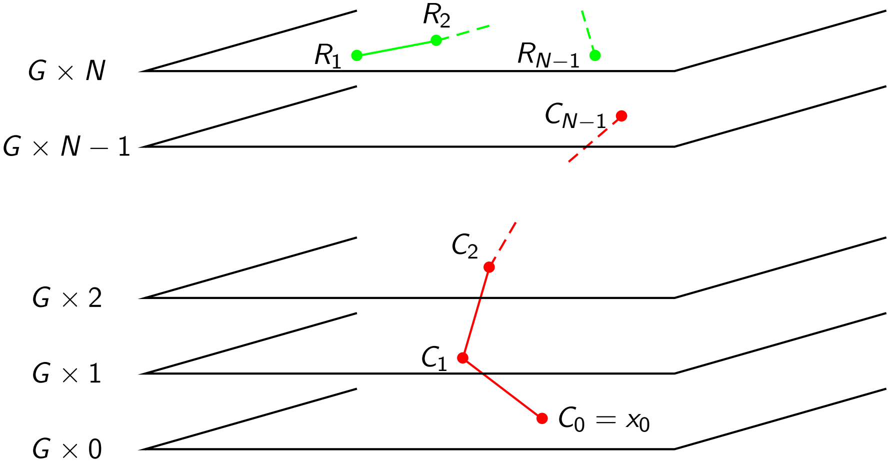

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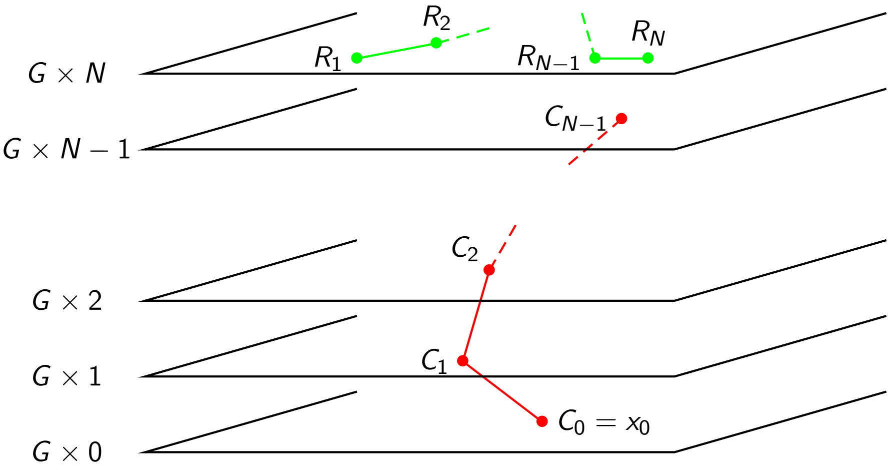
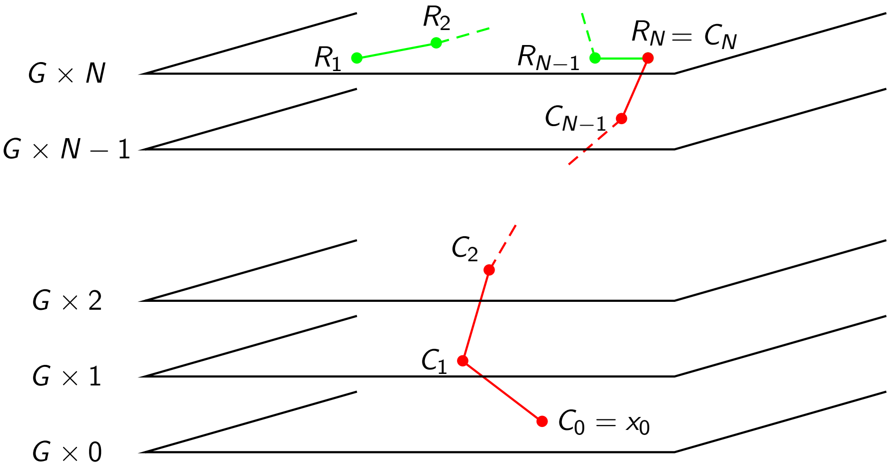


$\mathcal{H} : G \times I_N \to G$ from $\mathcal{H}_0 = C_{x_0}$ to $\mathcal{H}_N = 1_G$ gives a **winning strategy** :

	R_1	R_2	$\dots$	R_{N-1}	R_N
C_0	C_1	C_2	$\dots$	C_{N-1}	C_N

with $C_i = \mathcal{H}_i(R_i)$, for $i = 1, \dots, N$.



1-dismantlability

A vertex $a \in V(G)$ is called 1-dismantlable if its open neighbourhood $N_G]a[:= N_G[a] - a$ is a 0-dismantlable graph. Then, we say that G is 1-dismantlable on $G - a$; notation : $G \searrow^1 G - a$.

A graph G is called 1-dismantlable if $V(G) = \{x_1, x_2, \ldots, x_n\}$ with

$$G = G_1 \searrow^1 G_2 \searrow^1 G_3 \ldots \searrow^1 G_i \searrow^1 G_{i-1} \searrow^1 \ldots \searrow^1 G_n = \{x_n\}$$

where $G_i := G[x_i, x_{i+1}, \ldots, x_n]$.

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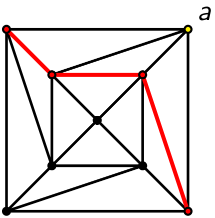
$\mathfrak{Q}_2$



a is 1-dismantlable

G is 1-dismantlable
and minimal in $D_1 \setminus D_0$
(in number of vertices)

k-dismantlability

Inductively, for k integer ≥ 1 :

A vertex $a \in V(G)$ is called k-dismantlable if its open neighbourhood $N_G]a[:= N_G[a] - a$ is a (k-1)-dismantlable graph. Then, we say that G is k-dismantlable on $G - a$; notation : $G \searrow^k G - a$.

A graph G is called k-dismantlable if $V(G) = \{x_1, x_2, \ldots, x_n\}$ with

$$G = G_1 \searrow^k G_2 \searrow^k G_3 \ldots \searrow^k G_i \searrow^k G_{i-1} \searrow^k \ldots \searrow^k G_n = \{x_n\}$$

where $G_i := G[x_i, x_{i+1}, \ldots, x_n]$.

NOTATIONS : $D_k := \{k\text{-dismantlable graphs}\}$

a strict hierachy

Theorem

The sequence $(D_k)_{k\geq 1}$ is strictly increasing and $D_\infty := \bigcup_{k\geq 0} D_k \subsetneqq D_{coll}$:

$$D_0 \subsetneqq D_1 \subsetneqq D_2 \subsetneqq \ldots\ldots \subsetneqq D_k \subsetneqq D_{k+1} \subsetneqq \ldots\ldots \subsetneqq D_{coll}$$

where D_{coll} is the set of graphs whos clique complex is collapsible.

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proof :

- For $k \geq 0$, $\mathfrak{Q}_{k+1} \in D_k \setminus D_{k-1}$ (cubions)
- For $n \geq 7$, $\Uparrow_n \in D_{coll} \setminus D_\infty := \bigcup_{k\geq 0} D_k$

The cubions $\mathfrak{Q}_n$, $n \in \mathbf{N}$

Definition of the *n-Cubion* $\mathfrak{Q}_n$

$V(\mathfrak{Q}_n) = \{\alpha_{i,\epsilon}, i = 1, \cdots, n \text{ and } \epsilon = 0,1\} \cup \{x = (x_1, \cdots, x_n), x_i = 0,1\}$ and $E(\mathfrak{Q}_n)$ defined by:

- $\forall i \neq j, \alpha_{i,\epsilon} \sim \alpha_{j,\epsilon'}$
- $\forall x \neq x', x \sim x'$
- $\forall i \in [n], \alpha_{i,1} \sim (x_1, \cdots, x_{i-1}, 1, x_{i+1}, \cdots, x_n)$ and

$\alpha_{i,0} \sim (x_1, \cdots, x_{i-1}, 0, x_{i+1}, \ldots, x_n)$

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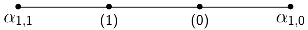
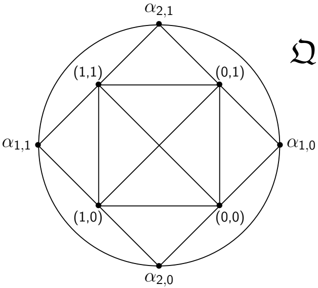


$\mathfrak{Q}_1$

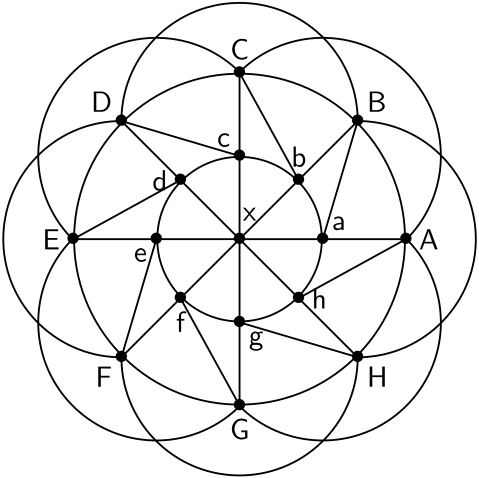


$\mathfrak{Q}_2$

$\mathfrak{Q}_n \in D_{n-1} \setminus D_{n-2}$, $n \in \mathbf{N}^*$ results from:

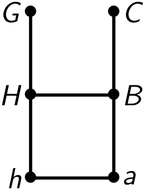
- $\forall\, i \in [n]$, $\forall \epsilon \in \{0, 1\}$, $\alpha_{i,\epsilon} \in D_{n-1}(\mathfrak{Q}_n) \setminus D_{n-2}(\mathfrak{Q}_n)$ because $N_{\mathfrak{Q}_n}(\alpha_{i,\epsilon}) \simeq \mathfrak{Q}_{n-1}$
- $\forall\, x$, $x \notin D_\infty(X)$ because $N_{\mathfrak{Q}_n}(x) \searrow^0 \overline{nK_2}$
- $\alpha_{2,\epsilon} \simeq\in D_{n-2}(\mathfrak{Q}_n - \{\alpha_{1,\epsilon}\}) \setminus D_{n-3}(\mathfrak{Q}_n - \{\alpha_{1,\epsilon}\})$
- ...
- $\alpha_{n-1,\epsilon} \in D_1(\mathfrak{Q}_n - \{\alpha_{i,\epsilon}, i = 1, \ldots, n-2\})$ $\setminus D_0(\mathfrak{Q}_n - \{\alpha_{i,\epsilon}, i = 1 \ldots, n-2\})$
- $\alpha_{n,\epsilon} \in D_0(\mathfrak{Q}_n - \{\alpha_{i,\epsilon}, i = 1 \ldots, n-1\})$
- So : $\mathfrak{Q}_n \searrow^{n-1} K_{2^n} \searrow^0 pt$

$\mathbb{T}_8 \in D_{coll} \setminus D_\infty$

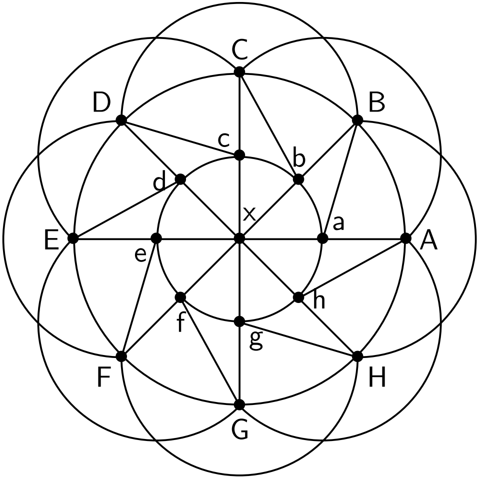


big-top $\mathbb{T}_8$

$\mathbb{T}_8 \in D_{coll} \setminus D_\infty$



$N_{\mathbb{T}_8}(A)$



big-top $\mathbb{T}_8$

and

$\Delta(\mathbb{T}_8) \searrow\searrow \Delta(\mathbb{T}_8)) - \{HB, HAB\}$

Back to evasiveness

a weak evasiveness conjecture

$$\left.\begin{array}{c} \exists\, k \in \mathbf{N},\ X \in D_k \\ X \ \textit{vertex transitive} \end{array}\right\} \Longrightarrow X \ \textit{complete}$$

Back to evasiveness

a weak evasiveness conjecture

$$\left.\begin{array}{c} \exists\, k \in \mathbf{N},\ X \in D_k \\ X \ \textit{vertex transitive} \end{array}\right\} \Longrightarrow X \ \textit{complete}$$

Proposition (case $k = 0$; E. F., B. Jouve)

Let X a finite graph.
If X is 0-dismantlable and vertex transitive, then X is a complete graph.

First direction of results

Let X a graph and k an integer ≥ 0.

Let $\mathcal{C}_k(X)$ denote the set of $(k+1)$-subsets of $V(X)$ which induce a complete subgraph of X (e.g. $\mathcal{C}_0(X) = V(X)$ and $\mathcal{C}_1(X) = E(X)$).

X will be called `k-transitive` if $\mathrm{Aut}(X)$ acts transitively on $\mathcal{C}_k(X)$, i.e.:

$$\forall\, (\{a_0, a_1, a_2, \ldots, a_k\}, \{b_0, b_1, b_2, \ldots, b_k\}) \in \mathcal{C}_k(X) \times \mathcal{C}_k(X),$$

$$\exists\, \varphi \in \mathrm{Aut}(X)\ s.t.\ \varphi(a_u) = b_u, \text{ for all } u \in \{0, 1, 2, \ldots, k\}$$

Exemples : Johnson graphs $J(v, k, i)$; for $i = 0$: Kneser graphs

Theorem

Let X a finite graph and k an integer ≥ 0.
If $X \in D_k$ and X is j-transitive for all $j \in \{0, 1, 2, \ldots, k\}$, then X is a complete graph.

Second direction of results

Theorem

If a Cayley graph $X = Cay(\mathbf{Z}/n\mathbf{Z}, S)$ is k-dismantlable for some integer $k \geq 0$, then X is a complete graph.
In particular, if a vertex transitive graph with a prime number of vertices is k-dismantlable for some integer $k \geq 0$, then it is a complete graph.

proof :

- By non evasiveness, $|\Delta X|^{\Gamma} \neq \emptyset$ (where $|\Delta X|^{\Gamma}$ the set of fixed points of $|\Delta X|$ under the action of Γ).
- By vertex transitivity, $V(X)$ is the unique orbit.
- So, X is a complete.

Thanks for your attention !